ZAMM · Z. Angew. Math. Mech. **88**, No. 2, 142 – 150 (2008) / **DOI** 10.1002/zamm.200700149

Antiplane shear deformations of an anisotropic elliptical inhomogeneity with imperfect or viscous interface

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Received 12 November 2007, revised 6 December 2007, accepted 11 December 2007
Published online 25 January 2008

Key words antiplane shear, elliptical inhomogeneity, imperfect bonding, viscous interface, anisotropic elasticity

Based on the Lekhnitskii-Eshelby approach of two-dimensional anisotropic elasticity, a semi-analytical solution is derived for the problem associated with an anisotropic elliptical inhomogeneity embedded in an infinite anisotropic matrix subjected to remote uniform antiplane shear stresses. In this research, the linear spring type imperfect bonding conditions are imposed on the inhomogeneity-matrix interface. We use a different approach than that developed by Shen et al. (2000) to expand the function encountered during the analysis for an imperfectly bonded interface. Our expansion method is in principle based on Isaac Newton's generalized binomial theorem. The solution is verified, both theoretically and numerically, by comparison with existing solution for a perfect interface. It is observed that the stress field inside an anisotropic elliptical inhomogeneity with a homogeneously imperfect interface is intrinsically nonuniform. The explicit expression of the nonuniform stress field within the inhomogeneity is presented. The nonuniform stress field inside the inhomogeneity is also graphically illustrated. A difference in internal stress distribution between a composite composed of anisotropic constituents and a composite composed of isotropic constituents is also observed. We finally extend the solution derived for a linear spring type imperfect interface to address an elliptical inhomogeneity with a viscous interface described by the linear law of rheology. It is observed that the stress field inside an elliptical inhomogeneity with a viscous interface is nonuniform and time-dependent.

1 Introduction

The antiplane shear (APS) deformation of anisotropic elastic materials is one of the simplest kinds of deformations in solid mechanics. In the following we outline some recent results within the context of APS. A comprehensive review for both linear and nonlinear elasticity was given by Horgan [1]. The decay rates of Saint-Venant end effects for sandwich strips and generally laminated anisotropic strips with imperfect bonding conditions were investigated by Baxter and Horgan [2] and Tullini et al. [3]. An elliptical inhomogeneity perfectly bonded to an unbounded matrix was considered by Kattis and Providas [4] by using the Lekhnitskii-Eshelby formulation and the so-called two-phase potentials. An anisotropic wedge of finite radius with traction-free condition on the circular segment and three different cases of boundary conditions on the radial edges was considered by Shahani [5]. The Green's function for an orthotropic quarter plane and a bimaterial that consists of two perfectly bonded orthotropic quarter planes was obtained by Ting [6] by employing the image singularity approach.

The objective of this research is to analyze the antiplane deformation of an anisotropic elliptical inhomogeneity imperfectly bonded to an infinite matrix. Here the condition of the imperfect interface is modeled to allow jumps in displacements proportional to the tractions at the interface [3, 7–16]. The isotropic counterpart of this problem has been considered by Shen et al. [13, 14]. In this investigation a semi-analytical solution to the present problem is derived based on the Lekhnitskii-Stroh formalism of two-dimensional anisotropic elasticity [4, 17]. Here we use a different approach than that developed by Shen et al. [13] to expand the function encountered during the analysis. Our expansion method is in principle based on Isaac Newton's generalized binomial theorem. It is observed that the stress field inside an anisotropic elliptical inhomogeneity with a homogeneously imperfect interface is intrinsically *nonuniform*, a phenomenon similar to that observed by Shen et al. [13] for a composite composed of isotropic constituents. A difference in internal stress distribution between a composite composed of anisotropic constituents and a composite composed of isotropic constituents is also observed in this study. We finally extend the solution derived for a linear spring imperfect interface to address an elliptical inhomogeneity with a viscous interface (or time-dependent sliding interface) described by the linear law of rheology [18–23]. It is observed that the stress field inside an elliptical inhomogeneity with a viscous interface is nonuniform and time-dependent.

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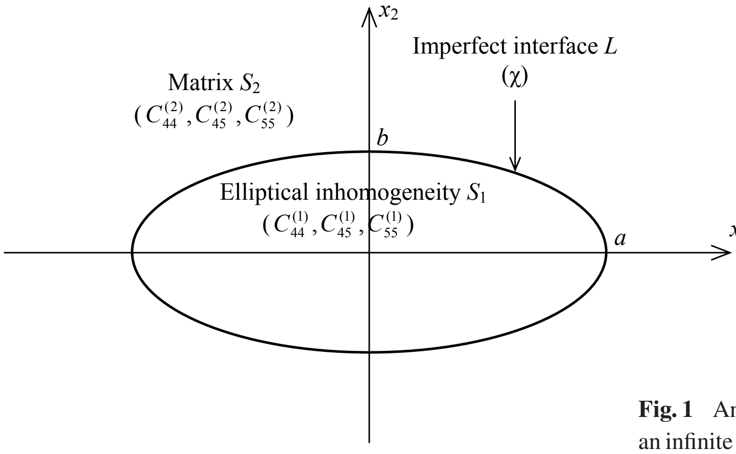

Fig. 1 An anisotropic elliptical inhomogeneity imperfectly bonded to an infinite matrix subjected to remote uniform antiplane shear stresses.

2 Basic equations

In a fixed rectangular coordinate system x_i, $(i = 1, 2, 3)$, let u_i and σ_{ij} be the displacement and stress, respectively. If the material possesses a symmetry plane $x_3 = 0$, then the stress-strain relation for an antiplane deformation is

$$\begin{aligned} \sigma_{31} &= C_{55}u_{,1} + C_{45}u_{,2}\,, \\ \sigma_{32} &= C_{44}u_{,2} + C_{45}u_{,1}\,, \end{aligned} \tag{1}$$

where $u = u_3, C_{44}, C_{45}, C_{55}$ are elastic constants and the comma stands for differentiation with x_i. The positive definiteness of the strain energy density will require that

$$C_{44} > 0\,, \qquad C_{55} > 0\,, \qquad C_{44}C_{55} - C_{45}^2 > 0\,. \tag{2}$$

For the special case of an orthotropic material with the orthotropy axes coinciding with the reference axes, one has $C_{45} = 0$. The equation of equilibrium is

$$\sigma_{31,1} + \sigma_{32,2} = C_{55}u_{,11} + 2C_{45}u_{,12} + C_{44}u_{,22} = 0\,. \tag{3}$$

The general solution of Eq. (3) can be expressed in terms of a single analytic function $f(z_p)$ as

$$u = \mathrm{Im}\{f(z_p)\}\,, \qquad z_p = x_1 + px_2 \tag{4}$$

where

$$p = \frac{-C_{45} + \mathrm{i}\sqrt{C_{44}C_{55} - C_{45}^2}}{C_{44}}\,. \tag{5}$$

The stresses σ_{31}, σ_{32} and the stress function ϕ are given by [4, 17]

$$\sigma_{31} + p\sigma_{32} = \mathrm{i}\lambda\,\mathrm{Im}\{p\}\overline{f'(z_p)}\,, \tag{6}$$

$$\phi = \lambda\,\mathrm{Re}\{f(z_p)\}\,, \tag{7}$$

where $\lambda = \sqrt{C_{44}C_{55} - C_{45}^2}$, and the stresses σ_{31}, σ_{32} are related to the stress function ϕ through

$$\sigma_{31} = -\phi_{,2}\,, \qquad \sigma_{32} = \phi_{,1}\,. \tag{8}$$

Let T be the antiplane surface traction component on a boundary L, if s is the arc-length measured along L such that, when facing the direction of increasing s, the material is on the right-hand side, it can be shown that [17]

$$T = \frac{d\phi}{ds}\,. \tag{9}$$

Consider now the antiplane deformation of an unbounded matrix containing an elliptical inhomogeneity. The linearly elastic materials occupying the inhomogeneity and the matrix are assumed to be homogeneous and anisotropic with associated elastic constants $C_{44}^{(1)}, C_{45}^{(1)}, C_{55}^{(1)}$ and $C_{44}^{(2)}, C_{45}^{(2)}, C_{55}^{(2)}$, respectively. We represent the matrix by the domain $S_2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \geq 1$ and assume that the inhomogeneity occupies the elliptical region $S_1 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$. The ellipse $L : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, whose semi-major and semi-minor axes are respectively a and b, will denote the inhomogeneity-matrix interface. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively. At infinity, the matrix is subject to remote uniform antiplane shear stresses σ_{31}^{∞} and σ_{32}^{∞}. On the imperfect interface L, the displacement jumps are proportional to the associated tractions. Thus the boundary conditions on the imperfect interface can be expressed as

$$\begin{aligned} &\phi_1 = \phi_2 \,, \\ &-\frac{d\phi_1}{ds} = T = \chi(u_2 - u_1)\,, \end{aligned} \qquad \text{on} \quad L \tag{10}$$

where χ is a non-negative imperfect interface constant parameter, and the increasing s is in the counterclockwise direction of the interface. The case where $\chi \to \infty$ corresponds to a perfect interface, while the case where $\chi \to 0$ corresponds to a traction-free surface.

Now consider the following mapping function [24]

$$z_1 = m_1(\zeta) = \tfrac{1}{2}(a - \mathrm{i}p_1 b)\zeta + \tfrac{1}{2}(a + \mathrm{i}p_1 b)\zeta^{-1}\,, \tag{11}$$

which can map an elliptical region with a cut in the $z_1(= x_1^{(1)} + \mathrm{i}x_2^{(1)} = x_1 + p_1 x_2)$-plane onto the annulus $\sqrt{|\rho|} \leq |\zeta| \leq 1$, $\left(\rho = \frac{a+\mathrm{i}p_1 b}{a-\mathrm{i}p_1 b}\right)$ in the ζ-plane. Next we consider another mapping function [24]

$$z_2 = m_2(\zeta) = \tfrac{1}{2}(a - \mathrm{i}p_2 b)\zeta + \tfrac{1}{2}(a + \mathrm{i}p_2 b)\zeta^{-1}\,, \tag{12}$$

which can map the outside of an elliptical region in the $z_2(= x_1^{(2)} + \mathrm{i}x_2^{(2)} = x_1 + p_2 x_2)$-plane onto the outside of the unit circle $|\zeta| \geq 1$ in the ζ-plane. For convenience, we write $f_1(z_1) = f_1(m_1(\zeta))$=$f_1(\zeta)$ and $f_2(z_2) = f_2(m_2(\zeta))$=$f_2(\zeta)$. In the following we endeavor to derive the expressions of $f_1(\zeta)$ and $f_2(\zeta)$.

3 General solution

In view of Eqs. (4), (7), (11) and (12), the above boundary conditions Eq. (10) can also be expressed in terms of $f_1(\zeta)$ and $f_2(\zeta)$ as follows

$$\begin{aligned} &\Gamma\left[f_1^+(\zeta) + \bar{f}_1^-(1/\zeta)\right] = f_2^-(\zeta) + \bar{f}_2^+(1/\zeta)\,, \\ &\zeta f_1'^+(\zeta) - \zeta^{-1}\bar{f}_1'^-(1/\zeta) = \gamma\sqrt{1 - h^2\cos^2\theta}\left[f_2^-(\zeta) - \bar{f}_2^+(1/\zeta) - f_1^+(\zeta) + \bar{f}_1^-(1/\zeta)\right]\,, \end{aligned} \qquad (\zeta = \mathrm{e}^{\mathrm{i}\theta}) \tag{13}$$

where $h = \sqrt{1 - b^2/a^2}$, $(0 \leq h < 1)$ is the eccentricity of the ellipse L, $\Gamma = \lambda_1/\lambda_2$ is a two-phase elastic parameter [4], and $\gamma = a\chi/\lambda_1$ is a dimensionless imperfect interface parameter.

Here $f_1(\zeta)$ can be expanded into the following form

$$f_1(\zeta) = \sum_{n=1}^{+\infty} a_n(\zeta^n + \rho^n\zeta^{-n})\,, \quad \left(\sqrt{|\rho|} \leq |\zeta| \leq 1\right) \tag{14}$$

where a_n, $(n = 1, 2, \ldots)$ are unknown constants to be determined.

It follows from Eq. $(13)_1$ that

$$f_2(\zeta) = k\zeta - \bar{k}\zeta^{-1} + \Gamma\sum_{n=1}^{+\infty}(a_n\rho^n + \bar{a}_n)\zeta^{-n}\,, \quad (|\zeta| \geq 1) \tag{15}$$

where the parameter k is related to the remote stresses σ_{31}^{∞} and σ_{32}^{∞} through

$$k = \frac{(\mathrm{i}a + p_2 b)(\sigma_{31}^{\infty} + \bar{p}_2\sigma_{32}^{\infty})}{2\lambda_2\,\mathrm{Im}\{p_2\}}\,. \tag{16}$$

Substituting the above expressions of $f_1(\zeta)$ and $f_2(\zeta)$ into Eq. $(13)_2$, we obtain

$$\begin{aligned} &\sum_{n=1}^{+\infty}\left[(na_n + n\bar{a}_n\bar{\rho}^n)\zeta^n - (na_n\rho^n + n\bar{a}_n)\zeta^{-n}\right] \\ &= \gamma\sqrt{1 - h^2\cos^2\theta}\left[\sum_{n=1}^{+\infty}\left[(\Gamma - 1)a_n\rho^n + (\Gamma + 1)\bar{a}_n - 2\bar{k}\delta_{n1}\right]\zeta^{-n} - \sum_{n=1}^{+\infty}\left[(\Gamma - 1)\bar{a}_n\bar{\rho}^n + (\Gamma + 1)a_n - 2k\delta_{n1}\right]\zeta^n\right]\,, \end{aligned} \tag{17}$$

$$(\zeta = \mathrm{e}^{i\theta})\,.$$

In order to solve the above equation, we first expand the function $\sqrt{1-h^2\cos^2\theta}$, $(0 \le h < 1)$ into the following convergent series form based on Newton's generalized binomial theorem as well as the binomial theorem in its normal sense

$$\sqrt{1-h^2\cos^2\theta} = I_0 + \sum_{n=1}^{+\infty} I_{2n}(\zeta^{2n} + \zeta^{-2n})\,, \qquad (\zeta = \mathrm{e}^{\mathrm{i}\theta}) \tag{18}$$

where

$$I_{2n} = \bar{I}_{2n} = \sum_{k=n}^{+\infty} (-1)^k \left(\tfrac{1}{2}\right)^{2k} C_{1/2}^{k} C_{2k}^{k-n} h^{2k}\,, \qquad (n = 0, 1, 2, \ldots) \tag{19}$$

with C_α^n the binomial coefficient defined by

$$C_\alpha^n = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\,. \tag{20}$$

Here it shall be stressed that we adopt a different approach than that developed by Shen et al. [13] to expand $\sqrt{1-h^2\cos^2\theta} = \frac{b}{a}\sqrt{1+b^*\sin^2\theta}$, $\left(b^* = \frac{a^2-b^2}{b^2}\right)$. One advantage of the present approach is that the integrals (A3) in Shen et al. [13], which are indispensable to the calculation of the expansion coefficients, can be circumvented.

Inserting the above expansion Eq. (18) into Eq. (17), and equating the coefficients for the same power of ζ, we finally obtain the following set of linear algebraic equations

$$\begin{aligned}
&a_{2n} = 0\,,\\
&(2n-1)a_{2n-1} + (2n-1)\bar{\rho}^{2n-1}\bar{a}_{2n-1} + \gamma\left[\sum_{m=1}^{+\infty} I_{2|n-m|} b_{2m-1} - \sum_{m=1}^{+\infty} I_{2(n+m-1)}\bar{b}_{2m-1}\right] = 0\,, \qquad (n \ge 1)\\
&b_{2n-1} = (\Gamma-1)\bar{a}_{2n-1}\bar{\rho}^{2n-1} + (\Gamma+1)a_{2n-1} - 2k\delta_{(2n-1)1}\,,
\end{aligned} \tag{21}$$

where $\delta_{(2n-1)1}$ is the Kronecker delta.

If we introduce three vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{J}$, two diagonal matrices $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$, and two real and symmetric matrices $\mathbf{A}$, $\mathbf{B}$ defined as

$$\mathbf{x}_1 = \begin{bmatrix} a_1 & a_3 & a_5 & \ldots \end{bmatrix}^T, \qquad \mathbf{x}_2 = \begin{bmatrix} b_1 & b_3 & b_5 & \ldots \end{bmatrix}^T, \qquad \mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & \ldots \end{bmatrix}^T, \tag{22a}$$

$$\mathbf{\Lambda}_1 = \mathrm{diag}\begin{bmatrix} 1 & 3 & 5 & \ldots \end{bmatrix}, \qquad \mathbf{\Lambda}_2 = \mathrm{diag}\begin{bmatrix} \bar{\rho} & \bar{\rho}^3 & \bar{\rho}^5 & \ldots \end{bmatrix}, \tag{22b}$$

$$\mathbf{A} = \mathbf{A}^T = \begin{bmatrix} I_0 & I_2 & I_4 & I_6 & \ldots\\ I_2 & I_0 & I_2 & I_4 & \ldots\\ I_4 & I_2 & I_0 & I_2 & \ldots\\ I_6 & I_4 & I_2 & I_0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \mathbf{B} = \mathbf{B}^T = \begin{bmatrix} I_2 & I_4 & I_6 & I_8 & \ldots\\ I_4 & I_6 & I_8 & I_{10} & \ldots\\ I_6 & I_8 & I_{10} & I_{12} & \ldots\\ I_8 & I_{10} & I_{12} & I_{14} & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{22c}$$

then Eq. $(21)_{2,3}$ can be concisely written into the following matrix forms

$$\begin{aligned}
&\mathbf{\Lambda}_1\mathbf{x}_1 + \mathbf{\Lambda}_1\mathbf{\Lambda}_2\bar{\mathbf{x}}_1 + \gamma\mathbf{A}\mathbf{x}_2 - \gamma\mathbf{B}\bar{\mathbf{x}}_2 = \mathbf{0}\,,\\
&\mathbf{x}_2 = (\Gamma-1)\mathbf{\Lambda}_2\bar{\mathbf{x}}_1 + (\Gamma+1)\mathbf{x}_1 - 2k\mathbf{J}\,.
\end{aligned} \tag{23}$$

Consequently $\mathbf{x}_1$ can be uniquely determined from the above set of linear algebraic equations as

$$\begin{bmatrix} \mathbf{x}_1\\ \bar{\mathbf{x}}_1 \end{bmatrix} = 2\begin{bmatrix} \gamma^{-1}\mathbf{\Lambda}_1 + (\Gamma+1)\mathbf{A} - (\Gamma-1)\mathbf{B}\bar{\mathbf{\Lambda}}_2 & \gamma^{-1}\mathbf{\Lambda}_1\mathbf{\Lambda}_2 + (\Gamma-1)\mathbf{A}\mathbf{\Lambda}_2 - (\Gamma+1)\mathbf{B}\\ \gamma^{-1}\mathbf{\Lambda}_1\bar{\mathbf{\Lambda}}_2 + (\Gamma-1)\mathbf{A}\bar{\mathbf{\Lambda}}_2 - (\Gamma+1)\mathbf{B} & \gamma^{-1}\mathbf{\Lambda}_1 + (\Gamma+1)\mathbf{A} - (\Gamma-1)\mathbf{B}\mathbf{\Lambda}_2 \end{bmatrix}^{-1}\begin{bmatrix} (k\mathbf{A} - \bar{k}\mathbf{B})\mathbf{J}\\ (\bar{k}\mathbf{A} - k\mathbf{B})\mathbf{J} \end{bmatrix}. \tag{24}$$

If the inhomogeneity is orthotropic, then $\mathbf{\Lambda}_2$ is real. As a result the above expression for $\mathbf{x}_1$ can be further simplified as

$$\mathbf{x}_1 = 2\,\mathrm{Re}\{k\}\left[\gamma^{-1}(\mathbf{A}-\mathbf{B})^{-1}\mathbf{\Lambda}_1(\mathbf{I}+\mathbf{\Lambda}_2)+(\Gamma+1)\mathbf{I}+(\Gamma-1)\mathbf{\Lambda}_2\right]^{-1}\mathbf{J} + 2\mathrm{i}\,\mathrm{Im}\{k\}\left[\gamma^{-1}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{\Lambda}_1(\mathbf{I}-\mathbf{\Lambda}_2)+(\Gamma+1)\mathbf{I}-(\Gamma-1)\mathbf{\Lambda}_2\right]^{-1}\mathbf{J}\,. \tag{25}$$

Particularly if the inhomogeneity is isotropic while the matrix is anisotropic and one is only concerned with the stress field inside the inhomogeneity, then it is equivalent to treat the matrix also as isotropic with shear modulus $\mu = \lambda_2$ subject to the virtual remote uniform shear stresses $\tilde{\sigma}_{31}^{\infty}$ and $\tilde{\sigma}_{32}^{\infty}$ such that

$$\tilde{\sigma}_{31}^{\infty} - \mathrm{i}\tilde{\sigma}_{32}^{\infty} = \frac{(a-\mathrm{i}p_2 b)(\sigma_{31}^{\infty}+\bar{p}_2\sigma_{32}^{\infty})}{(a+b)\,\mathrm{Im}\{p_2\}}\,. \tag{26}$$

4 Nonuniform stress field inside the elliptical inhomogeneity

If we truncate $\mathbf{x}_1$ to $n = N$, then $f_1(z_1)$ within the inhomogeneity can be expressed as

$$f_1(z_1) = \sum_{n=1}^{N} d_{2n-1} z_1^{2n-1}\,, \quad \left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\le 1\right) \tag{27}$$

where d_{2n-1}, $(n = 1,2,\ldots,N)$ are unknown constants to be determined.

In view of the fact that $f_1(z_1) = f_1(m_1(\zeta))=f_1(\zeta)$, then we arrive at the following set of linear algebraic equations

$$\sum_{n=m}^{N}\left(\frac{a-ip_1 b}{2}\right)^{2n-1}\rho^{n-m}C_{2n-1}^{n-m}d_{2n-1} = a_{2m-1}\,, \qquad (m=1,2,\ldots,N)\,. \tag{28}$$

Consequently, the vector $\mathbf{x}_3 = \begin{bmatrix} d_1 & d_3 & \ldots & d_{2N-1}\end{bmatrix}^T$ can be uniquely determined to be

$$\mathbf{x}_3 = \mathbf{C}^{-1}\mathbf{x}_1\,, \tag{29}$$

where the upper triangular matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} \left(\frac{a-\mathrm{i}p_1 b}{2}\right)\rho^0 C_1^0 & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^3\rho^1 C_3^1 & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^5\rho^2 C_5^2 & \ldots & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^{2N-1}\rho^{N-1}C_{2N-1}^{N-1} \\ 0 & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^3\rho^0 C_3^0 & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^5\rho^1 C_5^1 & \ldots & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^{2N-1}\rho^{N-2}C_{2N-1}^{N-2} \\ 0 & 0 & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^5\rho^0 C_5^0 & \ldots & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^{2N-1}\rho^{N-3}C_{2N-1}^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \left(\frac{a-\mathrm{i}p_1 b}{2}\right)^{2N-1}\rho^0 C_{2N-1}^0 \end{bmatrix}. \tag{30}$$

Now that the stress field inside the inhomogeneity can be explicitly given by

$$\sigma_{31}+\bar{p}_1\sigma_{32} = -\mathrm{i}\lambda_1\,\mathrm{Im}\{p_1\}\sum_{n=1}^{N} d_{2n-1}(2n-1)(x_1+p_1x_2)^{2n-2}\,, \qquad \left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\le 1\right). \tag{31}$$

It is observed from the above expression that the stress field inside an anisotropic elliptical inhomogeneity with a homogeneously imperfect interface is intrinsically *nonuniform*.

Next we consider two special cases in which uniform internal stress field exists

(i) A perfect interface, i.e., $\gamma\to\infty$. In this case it follows from Eq. (21) that

$$a_1 = \frac{2(\Gamma+1)k+2\bar{\rho}(1-\Gamma)\bar{k}}{(\Gamma+1)^2-(\Gamma-1)^2\,|\rho|^2}\,, \qquad a_n = 0\,, \quad (n=2,3,\ldots,+\infty) \tag{32}$$

Consequently, the stress field, which is uniform inside the perfectly bonded elliptical inhomogeneity, is given by

$$\sigma_{31}+p_1\sigma_{32} = \frac{2\mathrm{i}\lambda_1\,\mathrm{Im}\{p_1\}\bar{a}_1}{a+\mathrm{i}\bar{p}_1 b}\,, \quad \left(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\le 1\right) \tag{33}$$

which is checked to be the result derived by Kattis and Providas [4, Eq. (36)] by using the two-phase potentials.

Note: We find that there is a typo in the expression of A_2 below Eq. (35) in [4], the correct one is given below

$$A_2 = (1+\Gamma)\frac{(1+\Gamma)\bar{M}_1^{\mathrm{R}} M_1^{\mathrm{L}} \bar{\sum}^{\infty} - (1-\Gamma)\bar{M}_1^{\mathrm{L}} \bar{M}_2^{\mathrm{R}} \sum^{\infty}}{(1+\Gamma)^2 \left|M_1^{\mathrm{R}}\right|^2 - (1-\Gamma)^2 \left|M_2^{\mathrm{R}}\right|^2} .$$

(ii) An imperfectly bonded anisotropic *circular* inhomogeneity ($a = b$ and $h = 0$). In this case it is observed from Eq. (18) that

$$I_{2n} = \delta_{n0} , \qquad (n = 0, 1, 2, \ldots, +\infty) . \tag{34}$$

Consequently it follows from Eq. (21) that

$$a_1 = \frac{2\gamma\left[1+\gamma(\Gamma+1)\right]k - 2\gamma\bar{\rho}\left[1+\gamma(\Gamma-1)\right]\bar{k}}{\left[1+\gamma(\Gamma+1)\right]^2 - |\rho|^2\left[1+\gamma(\Gamma-1)\right]^2} , \qquad a_n = 0, \quad (n = 2, 3, \ldots, +\infty) \tag{35}$$

with $\rho = (1 + \mathrm{i}p_1)/(1 - \mathrm{i}p_1)$ and $k = a(\mathrm{i} + p_2)(\sigma_{31}^{\infty} + \bar{p}_2\sigma_{32}^{\infty})/[2\lambda_2 \operatorname{Im}\{p_2\}]$.

We observe that the stress field within an imperfectly bonded anisotropic circular inhomogeneity is still *uniform*, and the uniform stress field is given by

$$\sigma_{31} + p_1\sigma_{32} = \frac{2\mathrm{i}\lambda_1 \operatorname{Im}\{p_1\}\bar{a}_1}{a(1 + \mathrm{i}\bar{p}_1)} , \quad \left(x_1^2 + x_2^2 \le a^2\right) \tag{36}$$

5 An illustrative example

As an illustration of the obtained solution, we consider a composite composed of a graphite-epoxy elliptical inhomogeneity imperfectly bonded to a wood matrix. The material properties of the inhomogeneity and the matrix are taken from [3] as

$$C_{44}^{(1)} = 2.75790\,\mathrm{Gpa} , \quad C_{45}^{(1)} = 0 , \quad C_{55}^{(1)} = 4.13685\,\mathrm{Gpa} ,$$
$$C_{44}^{(2)} = 0.05\,\mathrm{Gpa} , \quad C_{45}^{(2)} = 0 , \quad C_{55}^{(2)} = 0.5\,\mathrm{Gpa} .$$

It is observed that in this example both the elliptical inhomogeneity and the surrounding matrix are orthotropic. Fig. 2 illustrates the two stress components σ_{31} and σ_{32} along the inhomogeneity side of the imperfect interface L for different values of the imperfect interface parameter γ when the remote stress is σ_{31}^{∞}, ($\sigma_{32}^{\infty} = 0$) and $a/b = 5$. Fig. 3 demonstrates the stress components σ_{31} and σ_{32} along the inhomogeneity side of the imperfect interface L for different values of the imperfect interface parameter γ when the remote stress is σ_{32}^{∞}, ($\sigma_{31}^{\infty} = 0$) and $a/b = 5$. Due to the fact that the graphite-epoxy

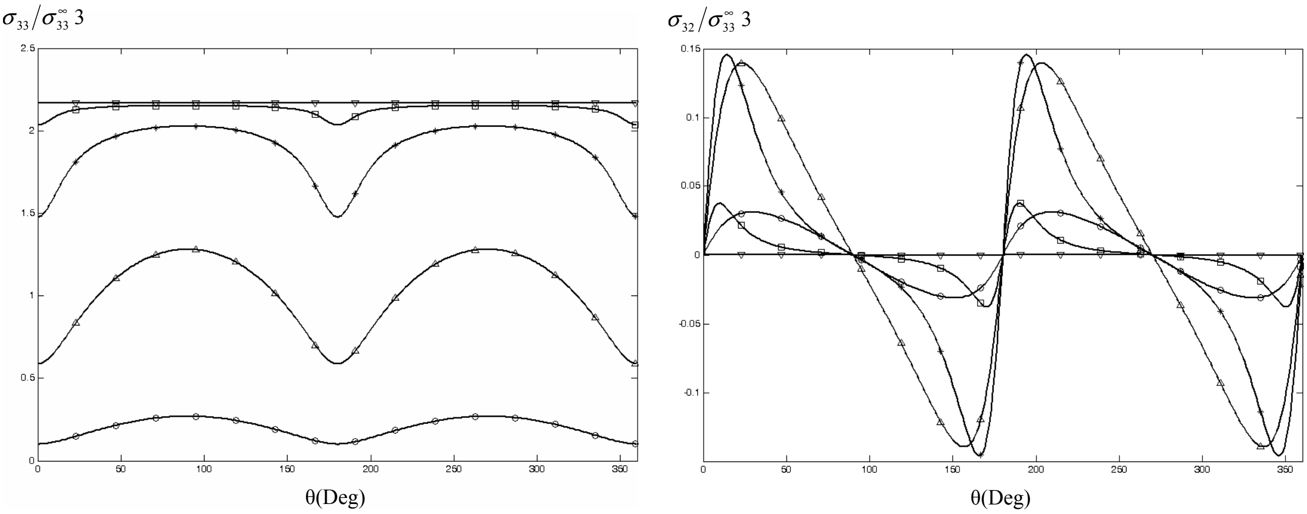

Fig. 2 Non-uniformity of stresses σ_{31} and σ_{32} along the interface when the remote stress is σ_{31}^{∞} with $a/b = 5$ (○— $\gamma = 0.01$, △— $\gamma = 0.1$, ∗— $\gamma = 1$, □— $\gamma = 10$, ▽— $\gamma = \infty$).

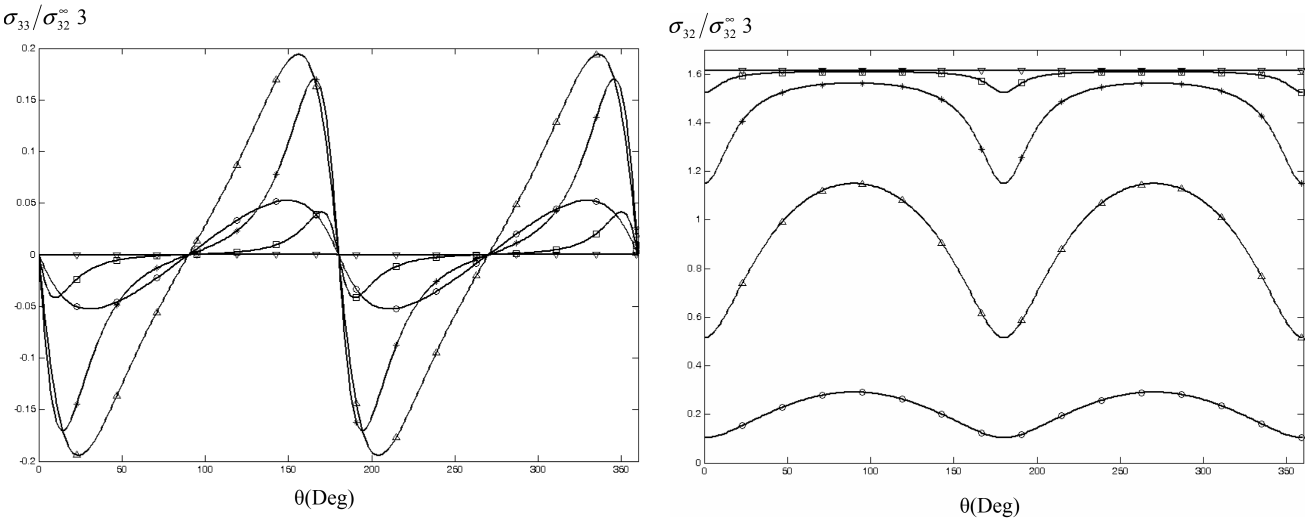

Fig. 3 Non-uniformity of stresses σ_{31} and σ_{32} along the interface when the remote stress is σ_{32}^{∞} with $a/b = 5$ ($\circ$— $\gamma = 0.01$, $\triangle$— $\gamma = 0.1$, $*$— $\gamma = 1$, $\square$— $\gamma = 10$, ∇— $\gamma = \infty$).

elliptical inhomogeneity is orthotropic, then we can adopt Eq. (25) to calculate $\mathbf{x}_1$. Furthermore we truncate $\mathbf{x}_1$ to $n = 20$ to ensure that the obtained results are sufficiently accurate with the relative errors being less than 0.1%. It is observed from Figs. 2 and 3 that the non-uniformity of the stresses σ_{31} and σ_{32} inside the imperfectly bonded inhomogeneity is apparent, especially when $\gamma = 0.1 \sim 1$. The observation of the nonuniform internal stress field is consistent with the result in [13]. When $\gamma \to \infty$ for a perfect interface, $\sigma_{31} = 2.1670\sigma_{31}^{\infty}$ and $\sigma_{32} = 1.6139\sigma_{32}^{\infty}$ within the inhomogeneity, just the results calculated by using Eqs. (32) and (33) (or using Eq. (36) in [4]). When the remote stress is σ_{31}^{∞}, it is observed from Fig. 2 that σ_{31} is an increasing function of the imperfect parameter γ and its value lies between zero for a totally debonded interface and the uniform value for a perfect interface, while σ_{32} can reach its maximum absolute value ($\approx 0.15\sigma_{31}^{\infty}$) when $\gamma = 1$. On the other hand when the remote stress is σ_{32}^{∞}, it is observed from Fig. 3 that σ_{32} is an increasing function of γ and its value lies between zero for a totally debonded interface and the uniform value for a perfect interface, while σ_{31} can reach its maximum absolute value ($\approx 0.2\sigma_{32}^{\infty}$) when $\gamma = 0.1$. The above phenomenon is quite different from that observed by Shen et al. [13] for a composite composed of isotropic constituents.

6 Extension to a viscous interface

The solution obtained above for a linear spring type imperfect interface can be conveniently extended to address an elliptical inhomogeneity with a viscous interface described by the following linear law of rheology [18–23]

$$\begin{aligned} &\phi_1 = \phi_2\,, \\ &-\frac{d\phi_1}{ds} = T = \eta(\dot{u}_2 - \dot{u}_1)\,, \end{aligned} \qquad \text{on} \quad L \tag{37}$$

where a dot over the quantity denotes differentiation with respect to the time t, and η is the interface slip constant which can be measured through properly designed experiment. It is implied in Eq. (37) that we have ignored the inertia effect for both the elliptical inhomogeneity and the surrounding matrix [18, 19, 22, 23, 25–28].

Now the two analytic functions defined in the inhomogeneity and the matrix are denoted as $f_1(\zeta, t)$ and $f_2(\zeta, t)$ as a result of the introduction of the viscous interface which exhibits the time effect. In addition the expressions of $f_1(\zeta, t)$ and $f_2(\zeta, t)$ are identical to Eqs. (14) and (15) except now that a_n, $(n = 1, 2, \ldots)$ are not constants but functions of the time due to the viscous interface. Through a similar derivation presented in Sect. 3, we finally find $a_{2n} = 0$, $(n = 1, 2, \ldots)$ and obtain the following set of equations

$$\begin{aligned} &\mathbf{\Lambda}_1\mathbf{x}_1 + \mathbf{\Lambda}_1\mathbf{\Lambda}_2\bar{\mathbf{x}}_1 + \delta\mathbf{A}\dot{\mathbf{x}}_2 - \delta\mathbf{B}\dot{\bar{\mathbf{x}}}_2 = \mathbf{0}\,, \\ &\mathbf{x}_2 = (\Gamma - 1)\mathbf{\Lambda}_2\bar{\mathbf{x}}_1 + (\Gamma + 1)\mathbf{x}_1\,, \end{aligned} \tag{38}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$, $\mathbf{A}$ and $\mathbf{B}$ have been defined by Eq. (22), and $\delta = \frac{a\eta}{\lambda_1}$. It shall be noticed that here $\mathbf{x}_1$ and $\mathbf{x}_2$ are not constant vectors but functions of the time t due to the introduction of the viscous interface.

Substituting Eq. $(38)_2$ into $(38)_1$, we finally arrive at the following state-space equation for the state variables $\mathbf{x}_1$ and $\bar{\mathbf{x}}_1$

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\bar{\mathbf{x}}}_1 \end{bmatrix} = \mathbf{E} \begin{bmatrix} \mathbf{x}_1 \\ \bar{\mathbf{x}}_1 \end{bmatrix} , \tag{39}$$

where the matrix $\mathbf{E}$ is given by

$$\mathbf{E} = -\frac{1}{\delta} \begin{bmatrix} (\Gamma+1)\mathbf{A} - (\Gamma-1)\mathbf{B}\bar{\mathbf{\Lambda}}_2 & (\Gamma-1)\mathbf{A}\mathbf{\Lambda}_2 - (\Gamma+1)\mathbf{B} \\ (\Gamma-1)\mathbf{A}\bar{\mathbf{\Lambda}}_2 - (\Gamma+1)\mathbf{B} & (\Gamma+1)\mathbf{A} - (\Gamma-1)\mathbf{B}\mathbf{\Lambda}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_1\mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_1\bar{\mathbf{\Lambda}}_2 & \mathbf{\Lambda}_1 \end{bmatrix} . \tag{40}$$

At the initial moment $t = 0$, the displacement across the interface has no time to have a jump due to the dashpot [18,25]. Therefore at $t = 0$ the interface is a perfect one. As a result it follows from Eq. (32) that the initial value of $\mathbf{x}_1$ is given by

$$\mathbf{x}_1(0) = \frac{2(\Gamma+1)k + 2\bar{\rho}(1-\Gamma)\bar{k}}{(\Gamma+1)^2 - (\Gamma-1)^2\,|\rho|^2}\mathbf{J} , \tag{41}$$

with $\mathbf{J}$ having been defined by Eq. (22a).

Now the solution to Eq. (39) can be concisely given by [22,29]

$$\begin{bmatrix} \mathbf{x}_1(t) \\ \bar{\mathbf{x}}_1(t) \end{bmatrix} = \exp(\mathbf{E}t) \begin{bmatrix} \dfrac{2(\Gamma+1)k + 2\bar{\rho}(1-\Gamma)\bar{k}}{(\Gamma+1)^2 - (\Gamma-1)^2\,|\rho|^2}\mathbf{J} \\ \dfrac{2\rho(1-\Gamma)k + 2(\Gamma+1)\bar{k}}{(\Gamma+1)^2 - (\Gamma-1)^2\,|\rho|^2}\mathbf{J} \end{bmatrix} . \tag{42}$$

Apparently when $t > 0$ the stress field inside the elliptical inhomogeneity with a viscous interface is nonuniform and time-dependent. As $t \to \infty$ the internal stress field will approach zero. This phenomenon is quite different from the uniform and time-dependent stress field within a circular inhomogeneity with a viscous interface [18, 19]. It is observed from the above derivations that the matrix form notations are also very effective in addressing an elliptical inhomogeneity with a viscous interface.

7 Conclusions

A general solution to the antiplane deformation problem of an anisotropic elliptical inhomogeneity imperfectly bonded to an unbounded anisotropic matrix subject to remote uniform shear stresses σ_{31}^{∞} and σ_{32}^{∞} is derived. In this investigation, we use a method different from that proposed by Shen et al. [13] to expand the function $\sqrt{1-h^2\cos^2\theta} = \frac{b}{a}\sqrt{1+b^*\sin^2\theta}$. Extremely concise matrix form expressions for the unknown coefficients a_n, $(n = 1, 2, \ldots)$ are presented in Eqs. (24) and (25). The results show that the imperfect parameter γ and the anisotropy of the inhomogeneity and the matrix exert a significant influence on the (nonuniform) stress field inside the elliptical inhomogeneity. The nonuniformity of the stress field inside the inhomogeneity is due to the assumption of uniformity of the imperfect interface parameter χ and the non-circular shape ($a \neq b$) of the elliptical inhomogeneity, while uniformity of stress field inside the inhomogeneity is possible for a special class of inhomogeneously imperfect interface [30]. We finally address an elliptical inhomogeneity with a viscous interface and find that the stress field inside the elliptical inhomogeneity is in general nonuniform and time-dependent. The case in which uniform eigenstrains are imposed on the inhomogeneity (transformation problem) [4] or a screw dislocation is located in the matrix or located in the elliptical inhomogeneity [4,31,32] can also be similarly addressed with no much added difficulties. Possible extensions of the present research include treating the interface as a viscoelastic one which is modeled by linear spring and dashpot [25–28]. It is expected that the obtained solution in this research can be further employed to predict the effective anisotropic moduli of the elliptical fiber reinforced composite with spring-type imperfect interface [33].

Acknowledgements The authors are indebted to a referee for his/her very helpful comments and suggestions. This work was supported in part by AFOSR FA9550-06-1-0317.

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